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HW 10

A ME 553
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1. Refer to HW-9, Problem 2.

   (c) Transform the equations from (a) to three equations using the following coordinates: x, y, and θ where x and y position the mass center of the system, and θ is the angle that vector $\mathbf{d} = \mathbf{r_1} - \mathbf{r_2}$ makes with the x-axis.

   Solution. In terms of the old coordinates,

$$\begin{aligned} x &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \\ y &= \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} \\ \tan\theta &= \frac{\Delta y}{\Delta x}, \end{aligned}$$

   where $\Delta y = y_2 - y_1$ and $\Delta x = x_2 - x_1$. Additionally, $\Delta x = l\cos\theta$ and $\Delta y = l\sin\theta$. Differentiating wrt time,

$$\begin{aligned} \Delta\dot{x} &= -l\sin\theta\dot{\theta} \\ \Delta\dot{y} &= l\cos\theta\dot{\theta}. \end{aligned}$$

   Putting all of this information together, we have the following

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & abl\sin\theta \\ 0 & 1 & adl\cos\theta \\ 1 & 0 & acl\sin\theta \\ 0 & 1 & ael\cos\theta \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix}.$$

   where the following constants are used,

$$\begin{aligned} a &= \frac{m_1 m_2}{m_1^2 - m_2^2} \\ b &= \frac{m_1 - m_2}{m_1} \\ c &= \frac{m_1^2 + m_1 m_2 - 2m_2^2}{m_1 m_2} \\ d &= \frac{m_1 + m_2}{m_1} \\ e &= \frac{m_1 + m_2}{m_2}. \end{aligned}$$

   Now, for the acceleration $\dot{\mathbf{v}} = \mathbf{B}\dot{\mathbf{w}} + \dot{\mathbf{B}}\mathbf{w}$,

$$\dot{\mathbf{B}}\mathbf{w} = \begin{pmatrix} abl\cos\theta\dot{\theta}^2 \\ -adl\sin\theta\dot{\theta}^2 \\ acl\cos\theta\dot{\theta}^2 \\ -ael\sin\theta\dot{\theta}^2 \end{pmatrix}$$

So, now the acceleration transformation becomes

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & abl\sin\theta \\ 0 & 1 & adl\cos\theta \\ 1 & 0 & acl\sin\theta \\ 0 & 1 & ael\cos\theta \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} abl\cos\theta\dot{\theta}^2 \\ -adl\sin\theta\dot{\theta}^2 \\ acl\cos\theta\dot{\theta}^2 \\ -ael\sin\theta\dot{\theta}^2 \end{pmatrix}.$$

Now, for the reduced equations of motion $B^T M B\dot{w} = B^T g - B^T M\dot{B}w$,

$$\mathbf{B}^T\mathbf{M}\mathbf{B} = \begin{pmatrix} m_1+m_2 & 0 & al(m_1b+m_2c)\sin\theta \\ 0 & m_1+m_2 & al(m_1d+m_2e)\cos\theta \\ al(m_1b+m_2c)\sin\theta & al(m_1d+m_2e)\cos\theta & a^2l^2\left((m_1b^2+m_2c^2)\sin^2\theta+(m_1d^2+m_2e^2)\cos^2\theta\right) \end{pmatrix}$$

$$\mathbf{B}^T\mathbf{g} = \begin{pmatrix} f_{1_x}+f_{2_x} \\ f_{1_y}+f_{2_y} \\ (f_{1_x}b+f_{2_x}c)l\sin\theta+(f_{1_y}d+f_{2_y}e)l\cos\theta \end{pmatrix}$$

$$\mathbf{B}^T\mathbf{M}\dot{\mathbf{B}}\mathbf{w} = \begin{pmatrix} al(m_1b+m_2c)\cos\theta\dot{\theta}^2 \\ -al(m_1d+m_2e)\sin\theta\dot{\theta}^2 \\ \frac{a^2l^2}{2}\left(m_1(b^2-d^2)+m_2(c^2-e^2)\right)\sin 2\theta\dot{\theta}^2 \end{pmatrix}$$

Plugging in the given numerical information,

$$\begin{pmatrix} 8 & 0 & 20\sin\theta \\ 0 & 8 & -30\cos\theta \\ 20\sin\theta & -30\cos\theta & 57.5+62.5\cos^2\theta \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} 1-20\cos\theta\dot{\theta}^2 \\ 1+30\sin\theta\dot{\theta}^2 \\ 31.25\sin 2\theta\dot{\theta}^2-0.75\sin\theta-5\cos\theta \end{pmatrix}$$

2. The figure shows the front-left steering and suspension system of a car. The bodies are numbered (1)-(7) and the joints are labeled as P (prismatic or translational), R (revolute), and S (spherical). Consider this system for joint-coordinate formulation.

   (a) Select an adequate set of joints to cut. A reasonable approach is to select a joint for cutting that has the fewest number of constraints. Also select the cut joints such that you do not end up with a branch that is too long (too many bodies) and one that is too short.

   Solution. Pick the spherical joints connecting bodies 3 and 7 and 2 and 3. This should ensure that the open networks of joints left are not too long or too short.

(b) For the constructed open-tree system, construct the $\mathbf{B}$ matrix in symbolic form.

Solution. Follow the joints from body i to j to get matrix $\mathbf{B}$ (Assume body 1 is connected to ground, in joint formulation $\mathbf{G}_{1,1}$,

$$\begin{pmatrix} \mathbf{G}_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{2,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S}_{3,3} & \mathbf{0} & \mathbf{R}_{3,5} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S}_{4,3} & \mathbf{R}_{4,4} & \mathbf{R}_{4,5} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R}_{5,5} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{P}_{6,6} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{P}_{7,6} & \mathbf{S}_{7,7} \end{pmatrix}$$

(c) Symbolically construct the Jacobian, $\mathbf{D}^{\otimes}$, for the cut joints in the body-coordinate formulation. For example, if you cut the spherical joint between bodies (2) and (3), its Jacobian can be expressed symbolically as $[\mathbf{D}^{\otimes}_{2,3}\mathbf{D}^{\otimes}_{3,2}]$.

Solution. The Jacobian can be symbolically developed

$$\begin{pmatrix} \mathbf{0} & \mathbf{D}^{\otimes}_{2,3} & \mathbf{D}^{\otimes}_{3,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}^{\otimes}_{3,7} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}^{\otimes}_{7,3} \end{pmatrix}$$

(d) Construct the $\mathbf{C}$ matrix symbolically from the product $\mathbf{D}^{\otimes}\mathbf{B}$.

Solution. $\mathbf{D}^{\otimes}\mathbf{B}$ is

$$\begin{pmatrix} \mathbf{0} & \mathbf{D}^{\otimes}_{2,3}\mathbf{R}_{2,2} & \mathbf{D}^{\otimes}_{3,2}\mathbf{S}_{3,3} & \mathbf{0} & \mathbf{D}^{\otimes}_{3,2}\mathbf{R}_{3,5} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}^{\otimes}_{3,7}\mathbf{S}_{3,3} & \mathbf{0} & \mathbf{D}^{\otimes}_{3,7}\mathbf{R}_{3,5} & \mathbf{D}^{\otimes}_{3,7}\mathbf{P}_{7,6} & \mathbf{D}^{\otimes}_{7,3}\mathbf{S}_{7,7} \end{pmatrix}$$

(e) Expand and simplify the elements of matrix $\mathbf{C}$. Use vectors such as $\mathbf{d}_{\otimes,4}$. Show these vectors on the figure (select your own joint reference points and body origins).

Solution.

(f) What is the dimension of $\mathbf{C}$? Are the rows independent? If not, how many rows are redundant?

Solution.